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Analysis of the singular sets of the Landau-Lifshitz system *

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1 Introduction

- **Physical model**

  In 1935, L.D. Landau and E.M. Lifshitz derived the following equation, the so called Landau-Lifshitz system, which describes evolution of spin fields in continuum ferrimagnetism (see [LL]).

  $$\partial_t u = -\alpha_1 u \times (u \times F_{eff}) + \alpha_2 u \times F_{eff},$$

  where $u = (u^1, u^2, u^3) : R^m \times R_+ \to S^2 \subset R^3$ is the spin field; "$\times$" denotes the vector cross product in R^3; $\alpha_1 > 0$ is a Gilbert damping constant, α_2 is a exchange constant, and F_{eff} is the effective field containing contributions from exchange interaction crystalline anisotropy, magneto-static self energy, external magnetic field, etc (see [LN]).

  In particular, taking $F_{eff} = \Delta u$, corresponding to the pure isotropic case and without external magnetic fields, Landau-Lifshitz equation reads

  $$\partial_t u = -\alpha_1 u \times (u \times \Delta u) + \alpha_2 u \times \Delta u.$$

  When $\alpha_1 = 0$, the system is called the Heisenberg system.

- **The equivalent equation**

  Using the following formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, and the fact that $|u| = 1$ implies $u\Delta u = -|\nabla u|^2$, we have the following equivalent equation

  $$\begin{cases} \lambda_1 \partial_t u - \lambda_2 u \times \partial_t u = \Delta u + |\nabla u|^2 u & (x,t) \in R^m \times R_+ \\ u(x,0) = u_0(x), & x \in R^m. \end{cases} \tag{1.1}$$

  where $\lambda_1 = \frac{\alpha_1}{\alpha_1^2+\alpha_2^2}$, $\lambda_2 = \frac{\alpha_2}{\alpha_1^2+\alpha_2^2}$, and $|u_0(x)| = 1$,

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- **Well known results**
  - **1-Dimension** A lot work contributed to the study of solutions of L-L system has been made by some physicists and mathematicians such as H.C.Fogedby [Fo], M.Laksmanan, K. Nakamura [LN], K.Nakamura, T.Sasada [NS], L.A.Takhtalian [T], J.Tjon, J.Wright [TW], Y.Zhou, B.Guo S.Tan [ZGT] and so on.
  - **2-Dimension:** Singular points in the finite time are finite, because of, well-known, the conformal in-variation of the energy in the 2-dimension.
  - **High-Dimension** The global existence of weak solutions for the equation has been established by F.Alouges A.Soyeur [AS](1992) and B.Guo, M.Hong [GA](1993).

- **The problems concerned and their difficulties**
  - **What is the regularity to (1.1)?** This is a basic problem to any nonlinear equations considered in the space of the generalized functions such as Sobolev space. Furthermore we hope to know the behavior of the solution at a singular point, that is, one ask what happen at a singular point?
  - **The characteristics of the equation (1.1)**
    - The landau-Lifshitz equation is a parabolic type equation with the natural increasing term $|\nabla u|^2$.
    - In appearance, (1.1) is similar to the heat flow of harmonic map into sphere (if $\alpha_2 = 0$), however there is an anti-symmetric term $u \times \partial_t u$. In other words, the Landau-Lifshitz system is a more general system, which contains in particularly the equations of harmonic map into sphere and its heat flow.
  - **The difficulties in getting regularity** The natural increasing term and the anti-symmetric term are both difficult to regularity (no existence). The classical methods can't be used for the first one, and the second one breaks down the monotone property which is a main tool to deal with the harmonic maps and its heat flows as known. We can define the **stationary condition** in an analogous way as in harmonic maps by R.Schoen, the so called the variation of domain.

    In the case of harmonic map and its heat flow, we have the following fact:

    $$\text{The stationary condition} \quad \Longrightarrow \quad \text{monotonicity.}$$

    But In our case, the stationary condition does not imply the monotonicity , we will see it in the following. Maybe this is the crucial reason as which up to now one could not to get the regularity of Landau-Lifshitz system.

    It is well known that the weak harmonic map, without monotone property, may be almost discontinuous in three dimension (see T. Riviera).

- **Our main results**
  - **Part one:** We prove the stationary weak solution is smooth except $\mathcal{H}^m_\rho$-zero set, i.e.,

$$\mathcal{H}^m_\rho(\mathrm{Sing(u)}) = 0.$$

    The following results largely depend on the regularity got in the first part.
  - **Part two:** Let u_k be a sequence of the stationary weak solutions of (1.1) with the initial data u_{k0} and $\int_{R^m} |\nabla u_{k0}|^2 \leq \Lambda$. For fixed t let Σ^t be the **blow up set** of the sequence, then we have
    1. Σ^t is rectifiable, i.e., almost C^1 smooth.
    2. Σ^t moves by quasi-mean curvature if u_k are the strong stationary weak solutions of (1.1) and $\lambda_1 \geq 2|\lambda_2|$.
  - **Part three:** Let (x_0, t_0) be a singular point of u, by scaling we obtain the two blow up formulas.

2 Stationary weak solutions

In this section, we introduce the notions of the stationary weak solutions of Landau-Lifshitz system, and show some generalized monotonicity inequalities.

DEFINITION 2.1 *$u(x,t) \in W^{1,2}(R^m \times R_+, S^2)$ is called a* **stationary** *weak solution of (1.1), if it is a weak solution of (1.1) and satisfies the following two assumptions:*

$$\int_{R^m} 2(\lambda_1 u_t - \lambda_2 u \times u_t)\zeta \cdot \nabla u - |\nabla u|^2 div\zeta + 2\partial_j u \partial_k u \partial_j \zeta^k = 0, \tag{2.1}$$

$$\begin{aligned}(\int_{R^m\times t_2} - \int_{R^m \times t_1})|\nabla u|^2 \vartheta dx \\ \leq -\int_{t_1}^{t_2}\int_{R^m}[2(\lambda_1 u_t^2 \vartheta + \nabla u \nabla \vartheta u_t) - |\nabla u|^2 \vartheta_t] dxdt,\end{aligned} \tag{2.2}$$

where $t_2 > t_1 > 0$, the functions ζ, ϑ are smooth, and $\vartheta \geq 0$

u is called a **strong stationary** *weak solution if the equality in (2.2) holds.*

REMARK 2.1 *If a weak solution u of (1.1) satisfies the stability hypothesis defined by the requirement that, similar to heat flow in [Fe],*

$$\begin{aligned}&\int_0^\infty \int_{R^m} (\lambda_1 u_t - \lambda_2 u \times u_t)\partial_\tau \hat{u}^\tau|_{\tau=0} \\ &+\partial_\tau^+ \int_0^\infty \int_{R^m} |\nabla \hat{u}^\tau|^2 dxdt|_{\tau=0} \leq 0\end{aligned} \tag{2.3}$$

holds for each family $\hat{u}^\tau$ of the domain variation defined by $\hat{u}^\tau = u(F_\tau(x,t))$, where $F_\tau = (x + \tau\tilde{\zeta}, t + \tau\tilde{\vartheta})$, $\tilde{\zeta}, \tilde{\vartheta}$ are smooth functions and $\tilde{\vartheta} \geq 0$. Then the assumptions (2.1) and (2.2) hold (see Proposition 7 and 8 in [Fe]).

Certainly smooth solution is strong stationary.

Notations:

- **Point Sets and Fundamental Solution**

$$z=(x,t)\in R^m\times R,\ z_0=(x_0,t_0);$$

$$B_r(x_0)=\{x\in R^m:\quad |x_0-x|<r\},\ S_r(t_0)=\left\{(x,t):\quad t=t_0-r^2\right\},$$

$$P_r(z_0)=\left\{(x,t)\in R^m\times R:|x-x_0|<r,|t-t_0|<r^2\right\},$$

$$T_r(z_0)=\left\{(x,t)\in R^m\times R:t_0-4r^2<t<t_0-r^2\right\}.$$

$$G_{z_0}=\frac{1}{[4\pi(t_0-t)]^{m/2}}\exp(-\frac{|x-x_0|^2\lambda_1}{4(t_0-t)}),\quad t<t_0,$$

is the standard fundamental solution of the backward heat equation $\partial_t G-\lambda_1^{-1}\Delta G=0$;

- **Functionals**

$$\Phi_{z_0}(R,u)=R^2\int_{S_R(t_0)}|\nabla u|^2G_{z_0}dx;\ \Psi_{z_0}(R,u)=\int_{T_R(t_0)}|\nabla u|^2G_{z_0}dxdt;$$

$$\Theta_{z_0}(R_1,R_2,u)=\frac{\lambda_2^2}{\lambda_1}\int_{t_0-R_2^2}^{t_0-R_1^2}\int_{R^m}u_t^2(t_0-t)G_{z_0}dxdt;$$

$$E(r,u,z)=r^{2-m}\int_{B_r(z)}|\nabla u|^2dy;\ I(r,u,z)=r^{2-m}\int_{B_r(x)}|\partial_t u|^2dy;$$

$$E_p(r,u,z)=r^{-m}\int_{P_r(z)}|\nabla u|^2dydt;\ \mathcal{E}(r,u,z)=E_p(r,u,z)+r^\beta.$$

The stationary condition

$$\Downarrow$$

- **Energy inequality**

$$\begin{aligned}&\int_0^T\int_{R^m}2\lambda_1u_t^2dxdt+\int_{R^m}|\nabla u|^2(x,T)dx\\ \le\ &\int_{R^m}|\nabla u_0|^2dx=E_0.\end{aligned}\tag{2.4}$$

- **Generalized monotone inequality I**

$$\begin{aligned}&\int_{t_0-R_2^2}^{t_0-R_1^2}\int_{R^m}\lambda_1[\frac{ru_r}{\sqrt{2(t_0-t)}}\\ &-\sqrt{2(t_0-t)}(u_t-\frac{\lambda_2}{2\lambda_1}u\times u_t)]^2G_{z_0}dxdt\\ \le\ &\Phi_{z_0}(R_2,u)-\Phi_{z_0}(R_1,u)+\Theta_{z_0}(R_1,R_2,u).\end{aligned}\tag{2.5}$$

The equality holds if and only if u is strong stationary weak solution, i.e. the equality in (2.2) (or (2.3))holds.

- **Generalized monotone inequality II**

  Suppose that $I(r,u,z) \le \frac{K}{\lambda_1^2+\lambda_2^2} r^{-\alpha}$ for $0 < r \le r_0$, where $K > 0, 0 < \alpha < 1/2$. Then $e^r(E(r,u,z) + \frac{K}{1-\alpha} r^{1-\alpha})$ is non-deceasing with respect to r, precisely

$$\begin{aligned} &\frac{d(e^r(E(r,u,z) + \frac{K}{1-\alpha} r^{1-\alpha}))}{dr} \\ \ge\ & 2e^r r^{2-m} \int_{\partial B_r} |\partial_r u|^2 d\sigma + \frac{Ke^r}{1-\alpha} r^{1-\alpha}. \end{aligned} \tag{2.6}$$

REMARK 2.2 *In the heat flow case,* $\Theta_{z_0}(R_1, R_2, u) = 0$ *since* $\alpha_2 = 0$. *So* $\Phi_{z_0}(R,u)$ *is increasing with respect to* R.

Our idea is to hope that $\Theta_{z_0}(R,u)$ **is small whenever** R **is small**, which enables us to define the following set Σ_β. For any fixed $1/2 > \alpha, \beta > 0$, and fixed constant c_0, we define

$$\Sigma_\beta = \left\{ z \in R^m \times R_+ : \begin{array}{c} \limsup_{R\to 0} R^{-(m-2+\beta)} \\ \int_{t-R^2}^{t+R^2} \int_{B_{R\sqrt{|\log R|}}(x)} u_t^2(x,t)dxdt = \infty \end{array} \right\}$$

$$S_\alpha(u,t) = \left\{ x \in R^m : \limsup_{r\to 0} r^\alpha I(r,z,u) \ge c_0 > 0 \right\}.$$

The following lemma is true (see [Liu] [LT]).

LEMMA 2.1

$$\mathcal{H}_\varrho^{m-2+\beta+\epsilon}(\Sigma_\beta) = 0,$$

where $\mathcal{H}_\varrho$ *denotes the parabolic metric Hausdorff measure, and* $\epsilon > 0$ *is any positive constant.*

If $\int_{R^m} |\partial_t u|^2 < \infty$, *then for any* $0 < \alpha < 1/2$, *we have*

$$\mathcal{H}^{m-2-\alpha}(S_\alpha(u,t) < \infty.$$

The following lemma enables our idea to become possible, which is the key part of getting regularity.

LEMMA 2.2 *Let* $z_0 \notin \Sigma_\beta$, *i.e. there exist constants* $C_0(z_0)$ *and* $R_0(z_0) > 0$, *such that whenever* $R_0 \ge R > 0$,

$$\frac{1}{R^{m-2+\beta}} \int_{t_0-R^2}^{t_0+R^2} \int_{B_{\sqrt{|\log R|}R}(x_0)} u_t^2(x,t)dx \le C_0.$$

Then for any $0 < R < R_1 \le R_0$, *there exists a constant* C_1, *depending only on* C_0, E_0, m, β *and* $\alpha_i, i = 1,2$, *such that*

$$\Theta_{z_0}(R, R_1, u) \le C_1 R_1^\beta.$$

REMARK 2.3 *If* $z_0 \notin \Sigma_\beta$, *then for large* $\sqrt{t_0} \geq R_1 \geq R_0$ *the same inequality holds from energy estimate (2.4).*

From the lemma above we have the following estimate for the normalized energy

PROPOSITION 2.1 *Let* $z_0 \notin \Sigma_\beta$, *corresponding* C_0, $R_0 > 0$, *and let* $R_2 < 1/3$, *and* $\mathcal{E}(R_2, u, z_0) \leq \epsilon_1$, *where* ϵ_1 *small enough (e.g.* $\epsilon_1 \leq \exp(-(80R_0^2)^{-1}\lambda_1)$*), then there exists a constant* C_3, *depending on* C_0, E_0, m, *and* β *and* $\alpha_i, i = 1, 2$, *such that for any* $0 < R < R_2$, *the following inequality*

$$\mathcal{E}(R, u, z_0) \leq C_3 |\log \mathcal{E}(R_2, u, z_0)|^{m/2} \mathcal{E}(R_2, u, z_0).$$

holds.

REMARK 2.4 *Let* $z_0 \notin \Sigma_\beta$. *Then*

$$\mathcal{E}(R, u, z) \leq C_2(R_0)$$

3 Partial regularity of the stationary solutions

We have

THEOREM 3.1 *Let* $u \in W^{1,2}(R^m \times R_+, S^2)$ *be a global stationary weak solution of the Landau-Lifshitz system (1.1) with* $E(u_0) < \infty$, *where* $E(u_0) = \int_{R^m} |\nabla u_0|^2 dV$. *Then singular set of u, Sing(u), is a closed set with* $\mathcal{H}_p^m(Sing(u)) = 0$. *More precisely,* $Sing(u) \subset \Sigma_\beta \cup \{z : \limsup_{r \to 0} \mathcal{E}(r, u, z) \geq \epsilon_0\}$ *and* $\mathcal{H}_p^{m-2+\beta+\alpha}(\Sigma_\beta) = 0, \mathcal{H}_p^m(\{z : \limsup_{r \to 0} \mathcal{E}(r, u, z) \geq \epsilon_0\}) = 0$ *for any* $1/2 > \beta, \alpha > 0$, *where* $\mathcal{H}_p$ *denotes the parabolic metric Hausdorff measure.*

As the usual blow-up argument, the key part to the proof of Theorem 3.1 is the following small energy decay lemma.

LEMMA 3.1 *There exist constants* $0 < \epsilon_0, \tau < 1$ *such that if* $\mathcal{E}(r, u, z) \leq \epsilon_0 \leq \epsilon_1 \leq 1/2$, *then*

$$\mathcal{E}(\tau r, u, z) \leq \frac{1}{2}\mathcal{E}(r, u, z) \tag{3.1}$$

for any $z \in R^m \times R_+$ *and* $z \notin \Sigma_\beta$, *and* $0 < r < \sqrt{t}$ *small enough.*

The compact Lemma 3.1 can be proved by using Proposition 2.1 and famous compensated compactness principle [CLMS] and Hélein's trick.

4 Analysis of the blow up sets

4.1 Problems

Let u_k be a sequence of stationary weak solutions of (1.1) with initial data $u_k(x,0)$ and $\int_{R^m} |\nabla u_k(x,0)|^2 \le \Lambda$. By the energy inequality we have

$$\begin{aligned} &\int_0^T \int_{R^m} 2\lambda_1 \partial_t u_k^2 dx dt + \int_{R^m} |\nabla u_k|^2(x,T) dx \\ \le &\int_{R^m} |\nabla u_k(x,0)|^2 dx = E_{k0} \le \Lambda. \end{aligned} \tag{4.1}$$

Therefore we may assume that $u_k \to u$ weakly in $W^{1,2}(R^m \times R_+, S^2)$. We set

$$\Sigma^t_{\epsilon_0} = \cap_{r>0} \left\{ x \in R^m | \liminf_{k\to\infty} \mathcal{E}(r, u_k, z) \ge \epsilon_0 \right\},$$

$$\Sigma^t_\beta = \left\{ x \in R^m | \begin{array}{c} \limsup_{r\to 0} \liminf_{k\to\infty} \frac{1}{r^{m-2+\beta}} \\ \int_{t-r^2}^{t+r^2} \int_{B_{r|\log r|^{1/2}}(x)} |\partial_t u_k|^2 = \infty \end{array} \right\},$$

where $\epsilon_0 > 0$ defined in Theorem 3.1 and $1/2 > \beta > 0$ are the fixed constants. We call the set $\Sigma^t = \Sigma^t_{\epsilon_0} \cup \Sigma^t_\beta$ **the blow up set** for the sequence u_k at t and $\Sigma = \cup_{0<t<\infty} \Sigma^t \times \{t\}$ **the total blow up set** for the sequence u_k.

The purpose in this part is to analyze the blow up set Σ^t. In view of the geometric measure theory we first hope to know that

- is Σ^t rectifiable?
- Furthermore, one ask how to move Σ^t with respect to t?
- What happens at a singular point?

4.2 Rectifiable

Define

$$\mathcal{T}_\infty = \left\{ t \in R_+ | \liminf_{k\to\infty} \int_{R^m} |\partial_t u_k|^2 = \infty \right\}.$$

Then we have

$$\mathcal{H}^1(\mathcal{T}_\infty) = 0;$$
$$\mathcal{H}^{m-2}(\Sigma^t_{\epsilon_0}) < \infty;$$

If $t \notin \mathcal{T}_\infty$ then for any $\epsilon > 0$, $\mathcal{H}^{m-4+\beta+\epsilon}(\Sigma^t_\beta) = 0$.

REMARK 4.1 *Here we can not expect that $\mathcal{H}^{m-2}(\Sigma^t_{\epsilon_0}) = 0$, as we do not know if the sequence $\{|\nabla u_k|^2\}$ is of the equivalent continuous in the sense of integration, or equivalently, strong convergence.*

The small energy regularity and (4.1) imply we may assume that

$$|\nabla u_k|^2(\cdot,t)dxdt \rightharpoonup |\nabla u|^2(\cdot,t)dxdt + \nu_t dt,$$

$$|\partial_t u_k|^2(\cdot,t)dxdt \rightharpoonup |\partial_t u|^2(\cdot,t)dxdt + \mu,$$

in the sense of measure as $k \to \infty$, where ν_t is a nonnegative Radon measure in R^m supported in Σ^t, μ is a nonnegative Radon measure in $R^m \times R_+$ supported in Σ.

We have from the following monotonicity (2.6).

LEMMA 4.1 *If* $\liminf_{k\to\infty} I(r,u_k,z) \leq \frac{r^{-\alpha}}{\lambda_1^2+\lambda_2^2}$ *for* $0 < r \leq r_0$, *then* $e^r(E(r,u,z) + r^{2-m}\nu_t(B_r(x)) + \frac{r^{1-\alpha}}{1-\alpha})$ *is non-deceasing. Precisely*

$$\frac{d}{dr}(e^r(E(r,u,z) + r^{2-m}\nu_t(B_r(x)) + \frac{r^{1-\alpha}}{1-\alpha})) \geq \frac{e^r r^{1-\alpha}}{1-\alpha}.$$

Now we present our main theorem in this section.

THEOREM 4.1 *Let* u_k *be a strong stationary weak solution of (1.1) with the initial energy* $E_{k0} \leq \Lambda$. *Then for almost every* $t \in R_+$, ν_t *is* $\mathcal{H}^{m-2}$*-rectifiable, therefore* Σ^t *is* $\mathcal{H}^{m-2}$ *-rectifiable.*

We obvious have the following properties for $\theta(x,t)$.

LEMMA 4.2 *For almost every* t, $\nu_t = \theta(x,t)\mathcal{H}^{m-2}\lfloor\Sigma^t$, *and* $\theta(x,t)$ *is upper semi-continuous in* $\Sigma^t/S_\alpha(t)$ *with* $C(\epsilon_0) \leq \theta(x,t) \leq C(\Lambda)$ *for* $\mathcal{H}^{m-2}$*-a.e.*$x \in \Sigma^t$, *and therefore* $\theta(x,t)$ *is* $\mathcal{H}^{m-2}$ *approximate continuous for* $\mathcal{H}^{m-2}$ *a.e.* $x \in \Sigma^t$.

4.3 Quasi-mean curvature flow

We know that the blow up sets of the heat flow for harmonic maps move by the mean curvature (see [LT]). In this section we will calculate the curvature of Σ^t at the point $t \notin \mathcal{T}_\infty$ and verify Σ^t moves by the quasi-mean curvature. The difference from the heat flows is that the blow up set Σ^t in Landau-Lifshitz case moves by the quasi-mean curvature defined in the following, no the mean curvature except $\alpha_2 = 0$.

Using the monotonicity inequality (2.6) we can obtain the following important lemmas.

LEMMA 4.3 *let* $T \in T\Sigma^t$, *where* $T\Sigma^t$ *is the tangent bundle on* Σ^t. *If* $t \notin \mathcal{T}_\infty$, *then we have*

$$\lim_{\epsilon\to 0}\liminf_{k\to\infty}\int_{B_\epsilon(\Sigma^t)} |\nabla_T u_k|^2 = 0.$$

Here and in the sequel we denote by

$$B_\epsilon(X) = \{x \in R^m | dist(x,X) < \epsilon\}.$$

LEMMA 4.4
$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\int_{B_\epsilon(\Sigma^t)} \partial_j u_k \partial_i u_k \partial_j \zeta^i = \frac{1}{2}\int_{\Sigma^t} div_{(\Sigma^t)^\perp}(\zeta)\,\nu_t.$$

We also define the induced Radon measures $\overrightarrow{w_t}$, $\overrightarrow{w_t^*}$,and $\overrightarrow{w_t^{*\perp}}$ on Σ^t by the following respectively,

$$\begin{aligned}&\lim_{k\to\infty} 2\int_{R^m}(\lambda_1\partial_t u_k - \lambda_2 u_k\times\partial_t u_k)\nabla u_k\zeta\\ =\ & 2\int_{R^m}(\lambda_1 u_t - \lambda_2 u\times u_t)\zeta\cdot\nabla u + 2\int_{\Sigma^t}\overrightarrow{w_t}\zeta\end{aligned}$$

$$\begin{aligned}&\lim_{k\to\infty}\int_{R^m}(\zeta\cdot\nabla u_k)\partial_t u_k\\ =\ & \int_{R^m}(\zeta\cdot\nabla u)u_t + \int_{\Sigma^t}\zeta\overrightarrow{w_t^*}\end{aligned}$$

$$\begin{aligned}&\lim_{k\to\infty}\int_{R^m}(u_k\times\partial_t u_k)(\zeta\cdot\nabla u_k)\\ =\ & \int_{R^m}(u\times\partial_t u)(\zeta\cdot\nabla u) + \int_{R^m}\zeta\cdot\overrightarrow{w_t^{*\perp}}\end{aligned}$$

where ζ is a smooth vector field with compact support in R^m, and $\overrightarrow{w_t} = \overrightarrow{w_t}^1 - \overrightarrow{w_t}^2$ and every component $\overrightarrow{w_t}^i, i = 1,2$ is a nonnegative Radon measure on Σ^t. Analogously for $\overrightarrow{w_t^*}$ and $\overrightarrow{w_t^{*\perp}}$. It is obvious that
$$\overrightarrow{w_t} = \lambda_1\overrightarrow{w_t^*} - \lambda_2\overrightarrow{w_t^{*\perp}}.$$

Then we obtain from the definition of the stationary weak solutions and the two lemmas above

THEOREM 4.2 *Let ζ be a smooth vector field with compact support in R^m. If $t\notin\mathcal{T}_\infty$, then*
$$\begin{aligned}&\int_{\Sigma^t} div_{\Sigma^t}(\zeta)\nu_t + \int_{R^m}|\nabla u|^2 div\zeta - 2\partial_j u\partial_k u\partial_j\zeta^k\\ =\ & 2\int_{R^m}(\lambda_1 u_t - \lambda_2 u\times u_t)\zeta\cdot\nabla u + 2\int_{\Sigma^t}\overrightarrow{w_t}\zeta.\end{aligned}$$

We also have the following theorem.

THEOREM 4.3 *Suppose that u_k is a stationary weak solution of (1.1). Then for any $\vartheta\in C_0^\infty(R^m, R_+)$, we have*
$$\begin{aligned}&\int_{R^m}\vartheta\nu_t - \int_{R^m}\vartheta\nu_{t_0} + \Big(\int_{R^m\times t} - \int_{R^m\times t_0}\Big)|\nabla u|^2\vartheta\\ \le\ & -2\int_{t_0}^t\int_{\Sigma^t}(\lambda_1\vartheta\mu + \nabla\vartheta\overrightarrow{w_t^*})\\ & -2\int_{t_0}^t\int_{R^m}(\lambda_1 u_t^2\vartheta + \nabla u\nabla\vartheta u_t).\end{aligned}$$

And if u_k is strong stationary weak solution of (1.1). Then the equality above holds.

We are in the position to introduce the curvature of Σ^t. Suppose that u_k are the strong stationary weak solutions, and the limiting map u is a strong stationary weak solution. We then have from Theorem 4.2 that, for any $t \notin \mathcal{T}_\infty$,

$$\int_{\Sigma^t} div_{\Sigma^t}(\zeta)\nu_t = 2\int_{\Sigma^t} \overrightarrow{w_t}\zeta = -\int_{\Sigma^t} \overrightarrow{H_t}\zeta\nu_t$$

where $\overrightarrow{H_t} = -2\frac{d\overrightarrow{w_t}}{d\nu_t}$. Clearly for $\mathcal{H}^{m-2}$-a.e. $x \in \Sigma^t$

$$\overrightarrow{H_t}(x) = -2\lim_{r\to 0}\lim_{k\to\infty}\frac{\int_{B_r(x)}(\lambda_1\partial_t u_k - \lambda_2 u_k \times \partial_t u_k)\nabla u_k}{\int_{B_r(x)}|\nabla u_k|^2}.$$

Applying Lemma 4.3 we see that

$$\overrightarrow{H_t}(x) \in S^\perp(x).$$

Where $S = S(x) = T_x\nu_t$. Thus we obtain the first variation of varifold V_{ν_t} :

$$\delta V_{\nu_t} = -\nu_t \lfloor \overrightarrow{H_t}.$$

And $\overrightarrow{H_t}$ is the **generalized mean curvature** of Σ^t (see 4.3 in [A]). We define the total variation $||\delta V_{\nu_t}||$ of δV_{ν_t} by the requirement that

$$||\delta V_{\nu_t}||(U) = \sup\{\delta V_{\nu_t}(\zeta) : \zeta \in T\Sigma^t, spt\zeta \subset U \text{and} \quad |\zeta| \le 1\}.$$

Then by the representation theorems (see 2.5 in [F])

$$\overrightarrow{H_t} = -\frac{d||\delta V_{\nu_t}||}{d\nu_t}\eta_t, \tag{4.2}$$

where $\eta_t(x) \in S^{m-1}$. We also define the varifolds $\delta^* V_{\nu_t}$ and $\delta^{*\perp} V_{\nu_t}$ respectively by

$$\delta^* V_{\nu_t}(\zeta) = \int_{\Sigma^t} 2\zeta\overrightarrow{w_t^*}\nu_t,$$

$$\delta^{*\perp} V_{\nu_t}(\zeta) = \int_{\Sigma^t} 2\zeta\overrightarrow{w_t^{*\perp}}\nu_t.$$

Then we have

$$||\delta^* V_{\nu_t}|| = ||\delta^{*\perp} V_{\nu_t}||, \tag{4.3}$$

$$|\overrightarrow{H_t^*}| = |\overrightarrow{H_t^{*\perp}}|, \tag{4.4}$$

$$\overrightarrow{H_t} = \lambda_1\overrightarrow{H_t^*} - \lambda_2\overrightarrow{H_t^{*\perp}}, \tag{4.5}$$

$$|\overrightarrow{H_t}| \le (\lambda_1 + |\lambda_2|)|\overrightarrow{H_t^*}|. \tag{4.6}$$

Evidently $\overrightarrow{H_t^*} = -2\frac{d\overrightarrow{w_t^*}}{d\nu_t}$ and $\overrightarrow{H_t^{*\perp}} = -2\frac{d\overrightarrow{w_t^{*\perp}}}{d\nu_t}$, and for $\mathcal{H}^{m-2}$-a.e. $x \in \Sigma^t$

$$\overrightarrow{H_t^*}(x) = -2\lim_{r\to 0}\lim_{k\to\infty}\frac{\int_{B_r(x)}\partial_t u_k \nabla u_k}{\int_{B_r(x)}|\nabla u_k|^2}.$$

$$\overrightarrow{H_t^{*\perp}}(x) = -2\lim_{r\to 0}\lim_{k\to\infty}\frac{\int_{B_r(x)}(u_k\times\partial_t u_k)\nabla u_k}{\int_{B_r(x)}|\nabla u_k|^2}.$$

Applying Lemma 4.3 again we have

$$\overrightarrow{H_t^{*}}(x), \overrightarrow{H_t^{*\perp}}(x) \in S^{\perp}(x).$$

$\overrightarrow{H_t^{*}}$ is called the **quasi-mean curvature** of Σ^t.

Now we introduce the Brakke's quantity $\mathcal{B}^*(\nu_t, \vartheta)$.

DEFINITION 4.1 *Let $\vartheta \in C_0^2(R^m, R_+)$. Define*

$$\mathcal{B}^*(\nu_t,\vartheta) = -\int_{R^m}\frac{\lambda_1}{2}\vartheta|\overrightarrow{H_t^*}|^2\nu_t + \int_{R^m}\nabla\vartheta\cdot S^{\perp} : \overrightarrow{H_t^*}\nu_t,$$

where $S = S(x) \equiv T_x\nu_t$ for H^{n-2} a.e. $x \in \{\vartheta > 0\}$.

DEFINITION 4.2 *We say $\{\nu_t\}_{t\geq 0}$ a generalized Brakke's motion provided that for a.e. $t \geq 0$ and all $\vartheta \in C_0^2(R^m, R_+)$,*

$$d_t^+\nu_t(\vartheta) \leq \mathcal{B}^*(\nu_t,\vartheta),$$

where we denote by

$$d_t^+ f = \limsup_{s\to t}\frac{f(s)-f(t)}{s-t}.$$

We also say the measure family $\{\nu_t\}_{t\geq 0}$ (or surface Σ^t equivalently) moving by the quasi-mean curvature in the case.

We have

THEOREM 4.4 *Suppose that u_k are the strong stationary weak solutions of (1.1) and the limiting map u is also a strong stationary weak solution of (1.1), then the blow up measure $\{\nu_t\}$ is a generalized Brakke's motion.*

The following theorem asserts that the singular set Σ^t of Landau-Lifshitz system is a quasi-mean curvature flow.

THEOREM 4.5 *Suppose that blow up set Σ^t of Landau-Lifshitz system (1.1) with $\lambda_1 \geq 2|\lambda_2|$ is a smooth family of sub-manifolds in R^m and assume that it is a generalized Brakke's flow in the sense of Theorem 4.4. Then Σ^t is a quasi-mean curvature flow.*

Proof. We write $\Sigma^t = F(\cdot,t)(M_0^{m-2})$, which is a parametric representation of Σ^t. Suppose that

$$\partial_t F(x,t) = \vec{Y}(x,t)$$

with $\vec{Y} \in T\Sigma^{t\perp}$. By the first variation of a varifold with respect to integrands (see 4.9 in [A] or 2.4 in [I]), we have

$$\delta V_{\nu_t}(\vartheta)(\vec{Y}) = d_t^+ \int_{\Sigma^t} \vartheta \nu_t = \int_{\Sigma^t} (-\vartheta \overrightarrow{H_t} + \nabla\vartheta \cdot S^\perp) \cdot \vec{Y} \nu_t, \tag{4.7}$$

where $\overrightarrow{H_t}$ defined in (4.2) is the mean curvature of Σ^t. On the other hand, by the Theorem 4.4, we have

$$d_t^+ \int_{\Sigma^t} \vartheta \nu_t \leq \int_{\Sigma^t} (-\frac{\lambda_1}{2} \vartheta |\overrightarrow{H_t^*}|^2 + \nabla\vartheta \cdot S^\perp \cdot \overrightarrow{H_t^*}) \nu_t. \tag{4.8}$$

Therefore we obtain that from (4.7) (4.8) and (4.5)

$$\int_{\Sigma^t} \vartheta [\lambda_1 \overrightarrow{H_t^*} \cdot (\vec{Y} - \frac{1}{2} \overrightarrow{H_t^*}) - \lambda_2 \overrightarrow{H_t^{*\perp}} \cdot \vec{Y}] \nu_t \geq \int_{\Sigma^t} \nabla\vartheta \cdot (\overrightarrow{H_t^*} - \vec{Y}) \nu_t.$$

Now since $\vartheta > 0$ is arbitrary and $|\overrightarrow{H_t^*} \cdot \overrightarrow{H_t^{*\perp}}| \leq |\overrightarrow{H_t^*}|^2$ and $\lambda_1 \geq 2|\lambda_2|$, we have to have $\overrightarrow{H_t^*} = \vec{Y}$, which is the desired result.

4.4 Blow up analysis at a singular point

Let $z_0 = (x_0, t_0) \notin \Sigma_\beta$ be a singular point of u such that

$$\lim_{r \to 0} \mathcal{E}(r, u, z_0) \geq \epsilon_0$$

by Theorem 3.1. Set $u_k(z) = u(x_0 + r_k x, t_0 + r_k^2 t)$ where $x \in R^m$ and $t \in R_-$, then u_k satisfies (1.1) and by scaling , for any $z \in R^m \times R_-$,

$$\mathcal{E}(r_k, u, z) = \int_{P_1(z)} |\nabla u_k|^2.$$

By Proposition 2.1 we see that for small r_k

$$\int_{P_1(z)} |\nabla u_k|^2 \leq C(R_0),$$

and from the energy inequality we have

$$\int_{P_{1/2}(z)} |\partial_t u_k|^2 \leq c \int_{P_1(z)} |\nabla u_k|^2 \leq C(R_0).$$

Denote , for fixed constant $\delta > 0$,

$$D_k = \left\{ z \in R^m \times R_- : \begin{array}{c} x_0 + r_k x \in B_\delta(x_0), \\ t_0 + r_k^2 t \in [t_0 - \delta^2, t_0 + \delta^2] \end{array} \right\},$$

then $D_k \to R^m \times R_-$ as $k \to \infty$, since $r_k \to 0$. There is therefore a subsequence (still denoted by r_k) $r_k \to 0$ such that $u_k(x,t) \to v(x,t)$ weakly in $W^{1,2}_{loc}(R^m \times R_-, S^2)$,

$$|\nabla u_k|^2(\cdot, t) dx dt \to |\nabla v|^2(\cdot, t) dx dt + \nu_t dt$$

$$|\partial_t u_k|^2(\cdot,t)dxdt \to |\partial_t v|^2(\cdot,t)dxdt + \mu$$

in the sense of measure as $k \to \infty$, where ν_t is a nonnegative Radon measure in R^m supported in Σ^t, μ is a nonnegative Radon measure in $R^m \times R_-$ supported in Σ, here and in the sequel we use the same notations and results in the sections above.

We have by scaling

$$\begin{aligned}
&\int_{t_0-(r_kR_2)^2}^{t_0-(r_kR_1)^2}\int_{R^m}\lambda_1[\frac{r\partial_r u}{\sqrt{2(t_0-t)}}\\
&-\sqrt{2(t_0-t)}(\partial_t u-\frac{\lambda_2}{2\lambda_1}u\times\partial_t u)]^2 G_{z_0}dxdt\\
\le\ &\Phi_{z_0}(r_kR_2,u)-\Phi_{z_0}(r_kR_1,u)+\Theta_{z_0}(r_kR_1,r_kR_2,u).
\end{aligned}$$

Since $z_0 \notin \Sigma_\beta$, we have $\Theta_{z_0}(r_kR_1,r_kR_2,u) \le C_1(r_kR_2)^\beta$ by Lemma 2.1. Then $\Theta_{z_0}(r_kR_1,r_kR_2,u) \to 0$ as $k \to \infty$. Obviously, for any $R > 0$, $\Phi_{z_0}(r_kR,u) \to \lambda_1^{-m/2}\theta(x_0,t_0)$ as $k \to \infty$. We thus obtain

$$\begin{aligned}
&\int_{-R_2^2}^{-R_1^2}\int_{R^m}\lambda_1[\frac{r\partial_r u_k}{\sqrt{2(-t)}}-\sqrt{2(-t)}(\partial_t u_k\\
&-\frac{\lambda_2}{2\lambda_1}u_k\times\partial_t u_k)]^2 G_0 dxdt \to 0
\end{aligned}$$

as $k \to \infty$. That is,

$$\lambda_1\partial_t u_k-\frac{\lambda_2}{2}(u_k\times\partial_t u_k)-\frac{\lambda_1 r\partial_r u_k}{2(-t)}\to 0 \tag{4.9}$$

strongly in $L^2(R^m\times[-R_2,-R_1],R^3)$ as $k\to\infty$. Furthermore

$$u_k\times\partial_t u_k-u_k\times\frac{x\cdot\nabla u_k}{2(-t)}\to 0 \tag{4.10}$$

strongly in $L^2(R^m\times[-R_2,-R_1],R^3)$. Applying the results in section 4.2 we obtain the blow up formulas

THEOREM 4.6 *If $t_0 \notin \mathcal{T}_\infty$, and $z_0=(x_0,t_0)\notin\Sigma_\beta$ is a blow up point, then we have two blow up formulas*

$$\begin{aligned}
&\int_{R^m}(|\nabla u|^2 div(\zeta)-2\partial_j v\partial_k v\partial_j\zeta^k)\\
&-2\int_{R^m}(\lambda_1\partial_t v-\lambda_2 v\times\partial_t v)\nabla v\zeta\\
=\ &\int_{\Sigma^t}\frac{1}{2(-t)}(\lambda_1 x_{\vec n}\cdot\zeta_{\vec n}-\frac{\lambda_2}{2}\sigma(x)|x_{\vec n}\times\zeta_{\vec n}|)\nu_t-\int_{\Sigma^t}div_{(\Sigma^t)}(\zeta)\nu_t,
\end{aligned}\tag{4.11}$$

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_{R^m} 2(\lambda_1 v_t^2\vartheta + \nabla v\nabla\vartheta v_t)dxdt \\
&+(\int_{R^m\times t_2} - \int_{R^m\times t_1})|\nabla v|^2\vartheta dx \\
= &\int_{\Sigma^t}\vartheta\nu_{t_1} - \int_{\Sigma^t}\vartheta\nu_{t_2} - \frac{\lambda_1^3}{4\lambda_1^2+\lambda_2^2}\int_{t_1}^{t_2}\int_{\Sigma^t}\frac{\vartheta|x_{\vec{n}}|^2}{t^2}\nu_t \\
&-\int_{t_1}^{t_2}\int_{\Sigma^t}\frac{1}{2(-t)}(\lambda_1 x_{\vec{n}}\cdot\zeta_{\vec{n}} - \frac{\lambda_2}{2}\sigma(x)|x_{\vec{n}}\times\zeta_{\vec{n}}|)\nu_t, \qquad (4.12)
\end{aligned}
$$

where $\zeta\in C_0^\infty(R^m,R^m)$ *and* $\vartheta\in C_0^\infty(R^m,R_+)$, $y_{\vec{n}}(x)$ *denotes the projective vector of the vector* $y(x)$ *on the normal plane of* Σ^t *at* x, *and* $\sigma(x)=\pm 1$ *is a sign function determined by some Radon measure.*

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